Homework 4

2-3

3a $T(1) = 2$ $\quad T(x) = 3 + 3x$ $\quad T(x^2) = 6x + 4x^2$

$[T]_\beta = \begin{pmatrix} 2 & 3 & 0 \\ 0 & 3 & 6 \\ 0 & 0 & 4 \end{pmatrix}$ $\quad U(1) = (1,0,1)$ $\quad U(x) = (1,0,-1)$

$U(x^2) = (0,1,0)$

$[U]_\beta^\gamma = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix}$ $\quad UT(1) = (2,0,2)$

$(UT)(x) = (6,0,0)$

$UT(x^2) = (6,4,-6)$

$[UT]_\beta^\gamma = \begin{pmatrix} 2 & 6 & 6 \\ 0 & 0 & 4 \\ 2 & 0 & -6 \end{pmatrix}$ Verify that this equals $[U]_\beta^\gamma [T]_\beta$

11 $T^2 = T_0 \iff R(T) \subseteq N(T)$

$\Rightarrow$: Let $v \in R(T)$ $\quad v = Tw$ some w $\quad Tv = T^2 w = 0$

$\therefore v \in N(T)$ $\quad \therefore R(T) \subseteq N(T)$

$\Leftarrow$ Let $v \in V$ $\quad \therefore Tv \in R(T)$ $\quad \therefore Tv \in N(T)$ $\quad \therefore T(Tv) = 0$

$\therefore T^2 = T_0$

12 (a) Suppose $Tv = Tv'$ $\quad \therefore UTv = UTv'$ But UT one-one

$\therefore v = v'$

(b) Let $z \in Z$ $\quad z = UTv$ some v since UT onto

$\therefore z = U(Tv)$ so U onto

(c) Will show UT one-one: $UT(v) = 0$. Since U one-one

$Tv = 0$ Since T one-one, $v = 0$ $\quad \therefore N(UT) = 0$ $\quad \therefore UT$ one-one

13 $(AB)_{ii} = \sum_{k=1}^n A_{ik} B_{ki}$ $\quad Tr(AB) = \sum_{i=1}^n \sum_{k=1}^n A_{ik} B_{ki}$

$= \sum_{i,k=1}^n A_{ik} B_{ki}$ $\quad Tr(BA) = \sum_{r=1}^n \sum_{s=1}^n B_{rs} A_{sr} =$

$\sum_{r,s=1}^n A_{sr} B_{rs}$ $\quad \therefore Tr(AB) = Tr(BA)$

2.4

2d If $T: V \to W$ is invertible, $\dim V = \dim W$

10 (a) $AB = I$, I invertible ($I^{-1} = I$) By #9, A & B invertible

(b) $AB = I$ $\qquad$ $ABB^{-1} = IB^{-1}$

$$AI = IB^{-1}$$

$$A = B^{-1}$$

(c) Given LTs $T : V \to W$, $S : W \to V$ with $ST = id$ where $\dim V = \dim W$. Then $Tv = Tv' \Rightarrow STv = STv' \Rightarrow v = v'$ so T is one-one $\therefore$ T is invertible.

Similarly S is onto so S is invertible.

16 $\Phi(A) = B^{-1}AB$ $\qquad$ Φ LT: $B^{-1}(A + A')B = B^{-1}AB + B^{-1}A'B$. $\quad B^{-1}(aA)B = a(B^{-1}AB)$. Next Φ one-one: Suppose $\Phi(A) = 0$ (the zero matrix) $\qquad B^{-1}AB = 0$

$$B(B^{-1}AB)B^{-1} = BOB^{-1}$$

$$A = IAI = 0 \qquad \therefore N(\Phi) = \{0\} \qquad \therefore \Phi \text{ one-one}$$

$\therefore$ Φ isomorphism

17 (b) Let $v_1, \dots, v_k$ be a basis for V_0. Show $Tv_1, \dots, Tv_k$ is a basis for TV_0

20 Did nullity part in class. The rank part follows. However the rank part can be proved directly as follows.

$$\begin{tikzcd} V \arrow[r, "T"] \arrow[d, "\phi_\alpha"'] & W \arrow[d, "\phi_\beta"] \\ F^n \arrow[r, "L_A"] & F^m \end{tikzcd}$$

Show ϕ_β restricted to $R(T)$ takes $R(T)$ into $R(L_A)$ giving a LT $\phi'' : R(T) \to R(L_A)$. Show ϕ'' is an isomorphism. One-one: Suppose $\phi'' w = 0$ for $w \in R(T)$ $\therefore$ $\phi_\beta w = 0$ $\therefore$ $w = 0$ since ϕ_β one-one

Onto: Given $x \in R(L_A)$ $\quad x = L_A y$ some $y \in F^n$ ~~By~~ But $y = \phi_\alpha v$ some $v \in V$ since ϕ_α onto $\quad x = L_A \phi_\alpha v = \phi_\beta(Tv) = \phi''(Tv)$ where $Tv \in RT$ $\therefore$ ϕ'' onto $\therefore$ ϕ'' isomorphism

$\therefore$ $R(T) \cong R(L_A)$ $\quad$ $\therefore$ Therefore their dimensions are equal

Rank T = Rank L_A.